Practice Questions concerning the new material:

1. The mass of cereal in a box has a standard bell curve with mean 16.5 ounces and variance 0.151 ounces^2

a) What is the distribution including the parameters?

This is a normal distribution
$\mu = 16.5$, $\sigma^2 = 0.151$

b) What is the probability that the box of cereal has a mass that is between 16 and 18 ounces?

$$P(16 < X < 18) = P\left(\frac{16-16.5}{\sqrt{0.151}} < Z < \frac{18-16.5}{\sqrt{0.151}}\right) = P(-1.29 < Z < 3.86)$$
$$= P(Z < 3.86) - P(Z < -1.29) = P(Z < 3.86) - (1 - P(Z < 1.29) = 1 - 1 + 0.9015 = 0.9015$$

c) What are the limits for the central 57% of the mass of the cereal?

$P(-z < Z < z) = P(Z < z) - P(Z < -z) = P(Z < z) - 1 + P(Z < z) = 2P(Z < z) - 1 = 0.57$
$2P(Z < z) = 1.57$
$P(Z < z) = 0.785$

OR

$$P(Z < z) = 0.57 + \frac{1-0.57}{2} = 0.785$$

$z = 0.79$
$x_L = \mu - \sigma z = 16.5 - (0.79)(0.151) = 16.381$
$x_U = \mu + \sigma z = 16.5 + (0.79)(0.151) = 16.619$

2. The distribution of the errors of two instruments, X and Y, in a laboratory are independent of each other. In addition, each of the distributions has a normal distribution. If X has a mean of 0.02 with a variance of 0.03 and Y has mean of 0.01 with a variance of 0.015, what is the joint distribution of the errors of the two instruments? (Hint: Think about how to prove if two variables are independent.)

$X \sim N(0.02, 0.03)$ $f_X(x) = \frac{1}{\sqrt{2\pi 0.03}} e^{-(x-0.02)^2/2(0.03)}$	$Y \sim N(0.01, 0.015)$ $f_Y(y) = \frac{1}{\sqrt{2\pi 0.015}} e^{-(y-0.01)^2/2(0.015)}$

$$f_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi 0.03}} \frac{1}{\sqrt{2\pi 0.015}} e^{-(x-0.02)^2/2(0.03)} e^{-(x-0.01)^2/2(0.015)}$$
$$= \frac{1}{2\pi\sqrt{(0.03)(0.015)}} e^{-[(x-0.02)^2/0.06+(y-0.01)^2/0.03]}$$

3. Let the joint probability density function of X and Y be given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{6}{5}(x+y^2) & if\ 0<x<1, 0<y<1 \\ 0 & else \end{cases}$$

a) What is the probability that X is less than Y?

$$P(X<Y) = \int_0^1 \int_0^y \frac{6}{5}(x+y^2)dxdy = \frac{6}{5}\int_0^1 \left[\frac{x^2}{2}+y^2x\right]\Big|_0^y dy = \frac{6}{5}\int_0^1 \left[\frac{y^2}{2}+y^3\right]dy = \frac{6}{5}\left[\frac{y^3}{2\cdot 3}+\frac{y^4}{4}\right]\Big|_0^1$$
$$= \frac{6}{5}\left(\frac{1}{6}+\frac{1}{4}\right) = \frac{6}{5}\left(\frac{5}{12}\right) = \frac{1}{2}$$

b) Calculate the marginal PDF of X.

$$f_X(x) = \int_0^1 \frac{6}{5}(x+y^2)dy = \frac{6}{5}\left[xy+\frac{y^3}{3}\right]\Big|_0^1 = \frac{6}{5}\left(x+\frac{1}{3}\right)$$

c) Are X and Y independent? Why or why not?

No, $f_{X,Y}(x,y)$ is not separatable.

d) Calculate the conditional PDF of Y given X = x.

$$f_{Y|X}(y|X=x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\frac{6}{5}(x+y^2)}{\frac{6}{5}\left(x+\frac{1}{3}\right)} = \frac{x+y^2}{x+\frac{1}{3}}$$

e) Determine the conditional expectation of Y for each possible value of X.

$$\mathbb{E}_{Y|X}(Y|X=x) = \int_0^1 y\left(\frac{x+y^2}{x+\frac{1}{3}}\right)dy = \frac{1}{x+\frac{1}{3}}\int_0^1 (xy+y^3)dy = \frac{1}{x+\frac{1}{3}}\left[\frac{xy^2}{2}+\frac{y^4}{4}\right]\Big|_0^1 = \frac{1}{x+\frac{1}{3}}\left(\frac{x}{2}+\frac{1}{4}\right)$$

f) Determine the conditional variance of Y when X = 0.2.

Var(Y|X = 0.2) = $\mathbb{E}$(Y^2|X=0.2) – ($\mathbb{E}$(Y|X=0.2))^2 = 0.5 – 0.65625^2 = 0.0693

$$\mathbb{E}_{Y|X}(Y|X=2) = \frac{\frac{x}{2}+\frac{1}{4}}{x+\frac{1}{3}} = \frac{\frac{0.2}{2}+\frac{1}{4}}{0.2+\frac{1}{3}} = 0.65625$$

$$f_{Y|X}(y|X=0.2) = \frac{0.2+y^2}{0.2+\frac{1}{3}} = 1.875(0.2+y^2)$$

$$\mathbb{E}(Y^2|X=x) = \int_0^1 y^2 1.875(0.2+y^2)dy = 1.875\int_0^1 (0.2y^2+y^4)dy = 1.875\left[\frac{0.2y^3}{3}+\frac{y^5}{5}\right]\Big|_0^1$$

$$= 1.875\left(\frac{0.2}{3}+\frac{1}{5}\right) = 1.875\left(\frac{4}{15}\right) = 0.5$$

4. On average, a light bulb works for 5 years.

a) What is the distribution, including parameters? (Note: work includes determining what the parameter is.)

Exponential

$$\lambda = \frac{1}{\mathbb{E}(X)} = \frac{1}{5}$$

b) What is the probability that the light bulb will last for more than 7 years?

P(X > 7) = 1 – P(X < 7) = 1 – 1 + e$^{-7/5}$ = 0.247

OR

$$P(X>7) = \int_7^\infty \frac{1}{5}e^{-\frac{x}{5}}dx = -e^{-\frac{x}{5}}\Big|_7^\infty = 0 + e^{-7/5} = 0.247$$

c) What is the variance for the length of time that the light bulb will last? (Please indicate from where you are obtaining your answer – this is the work.)

$$Var(X) = \frac{1}{\lambda^2} = \frac{1}{1/25} = (\mathbb{E}(X))^2 = 25$$